Inequality 6

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Q) Suppose the polynomial ax^2+bx+c satisfies the following:
$a>0$, $a+b+c \geq 0$, $a-b+c \geq 0$, $a-c \geq 0$, $b^2-4ac \geq 0$. Prove that the roots are real and belong to $-1 \leq x \leq 1$.

Ans:- $ax^2+bx+c=0$

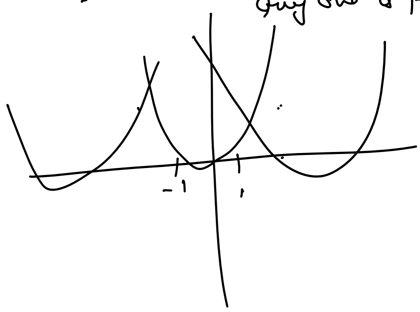
$$\text{Roots} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a-b+c \geq 0$ $\quad a-c \geq 0$

$\Rightarrow a+c \geq b$ $\quad \Rightarrow a \geq c$ $\quad \Rightarrow -a \leq -c$

$\Rightarrow -a-c \leq -b$

Idea



$a>0$

$P(x) = ax^2+bx+c$

$P(1) = a+b+c \geq 0$

$P(-1) = a-b+c \geq 0$

$$P\left(-\frac{b}{2a}\right) = \min(P(x)) = \frac{4ac-b^2}{4a} \leq 0$$

$P(x_1)=0$ $\quad P(x_2)=0$ i.e, x_1 and x_2 are roots of $P(x)$

$-\frac{b}{a} = x_1+x_2 \Rightarrow -(x_1+x_2) = \frac{b}{a}$

$\frac{c}{a} = x_1x_2$

$$\frac{a+b+c}{a} = -x_1-x_2+x_1x_2+1 = (1-x_1)(1-x_2) \geq 0$$

$$\frac{a-b+c}{a} = x_1+x_2+x_1x_2+1 = (1+x_1)(1+x_2) \geq 0$$

$$\frac{a-c}{a} = 1-x_1x_2 \geq 0$$

$(1-x_1)(1-x_2) \geq 0$

Either both +ve or both negative or any one 0.

$\underline{x_1x_2 \leq 1}$

If $x_2 = 0$,

$(1 - x_1) \geq 0 \quad x_1 \leq 1 \quad \Rightarrow \quad -1 \leq x_1, x_2 \leq 1$

$(1 + x_1) \geq 0 \quad x_1 \geq -1$

If $(1-x_1) \geq 0$ and $(1-x_2) \geq 0$ and $(1+x_1) \geq 0$ and $(1+x_2) \geq 0$

$x_1 \leq 1 \qquad x_2 \leq 1 \qquad x_1 \geq -1 \qquad x_2 \geq -1$

If $(1-x_1) \geq 0$ and $(1-x_2) \geq 0$ and $(1+x_1) \leq 0$ and $(1+x_2) \leq 0$ So this case is not valid

$x_1 \leq 1 \qquad x_2 \leq 1 \qquad x_1 \leq -1 \qquad x_2 \leq -1$

$\Rightarrow x_1 x_2 \geq 1 \quad \Rightarrow \Leftarrow$

If $(1-x_1) \leq 0$ and $(1-x_2) \leq 0$ and $(1+x_1) \geq 0$ and $(1+x_2) \geq 0$

$x_1 \geq 1 \qquad x_2 \geq 1 \qquad x_1 \geq -1 \qquad x_2 \geq -1$

So this case not possible.

$x_1 x_2 \geq 1 \quad \Rightarrow \Leftarrow$

So we get $-1 \leq x_1, x_2 \leq 1$

AM, GM Inequality :—

a, b be two <u>non-negative</u> real numbers. Then,

$$\boxed{\frac{a+b}{2} \geq \sqrt{ab}}$$

This is the Arithmetic Mean of a and b

This is Geometric Mean of a and b

* Equality holds when $a = b$

$$\frac{a+b}{2} = \sqrt{ab}$$

$$\Leftrightarrow (a+b)^2 = 4ab$$

$$(a+b)^2 - 4ab = 0$$

$$\Leftrightarrow \quad (a+b) = 4ab$$

$$\Leftrightarrow \quad (a+b)^2 - 4ab = 0$$

$$\Leftrightarrow \quad (a-b)^2 = 0$$

$$\Leftrightarrow \quad a = b$$

<u>Proof :-</u> $\dfrac{a+b}{2} - \sqrt{ab}$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{a}\sqrt{b}}{2}$$

$$= \left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right)^2 \geq 0$$

$$\Rightarrow \quad \frac{a+b}{2} - \sqrt{ab} \geq 0 \quad \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$$

Q> For $x \geq 0$, prove that $1+x \geq 2\sqrt{x}$

Ans :- $\dfrac{1+x}{2} \geq \sqrt{1 \cdot x}$ (AM-GM)

$\Rightarrow 1+x \geq 2\sqrt{x}$

Q> For $x > 0$, prove that $x + \frac{1}{x} \geq 2$

Ans:- $\dfrac{x+\frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} \Rightarrow x + \frac{1}{x} \geq 2$

Q> For $x, y \in \mathbb{R}^+$, prove that $x^2 + y^2 \geq 2xy$

Ans:- $\dfrac{x^2+y^2}{2} \geq \sqrt{x^2y^2} \Rightarrow x^2+y^2 \geq 2xy$

Q> For $x, y \in \mathbb{R}^+$, prove that, $2(x^2+y^2) \geq (x+y)^2$

$\ldots^2 \geq 2xy \Rightarrow 2(x^2+y^2) \geq x^2+2xy+y^2 = (x+y)^2$

Q> For $x, y \in \mathbb{R}$, prove that, $2(x^2+y^2) \geq (x+y)^2$

Ans:- $x^2+y^2 \geq 2xy \Rightarrow 2(x^2+y^2) \geq x^2+2xy+y^2 = (x+y)^2$

Q> For $x, y \in \mathbb{R}^+$, prove that, $\frac{1}{x}+\frac{1}{y} \geq \frac{4}{x+y}$

Ans:- $\frac{1}{x}+\frac{1}{y} = \frac{x+y}{xy} \geq \frac{4}{x+y} \Leftarrow (x+y)^2 \geq 4xy \Leftarrow (x-y)^2 \geq 0$

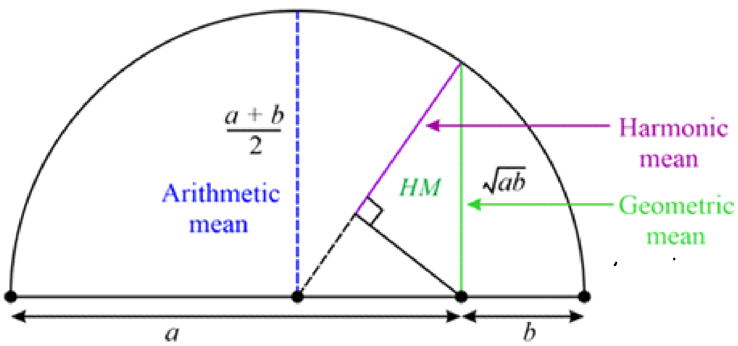
Q> If $0 < b \leq a$ then, $\frac{1}{8}\left(\frac{(a-b)^2}{a}\right) \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8}\left(\frac{(a-b)^2}{b}\right)$

Ans:- HomeWork

Harmonic Mean (HM):- $a, b \in \mathbb{R}^+$ has HM as $\frac{2}{\frac{1}{a}+\frac{1}{b}}$

$$AM \geq GM \geq HM$$

$$\frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$$



$GM \geq HM$

Proof:- $\sqrt{ab} - \frac{2}{\frac{1}{a}+\frac{1}{b}}$

$$= \sqrt{ab} - \frac{2ab}{a+b}$$

$$\frac{(a+b)\sqrt{ab} - 2ab}{}$$

$$= \frac{(a+b)\sqrt{ab} - 2ab}{a+b}$$

$$= \frac{a\sqrt{ab} + b\sqrt{ab} - 2ab}{a+b}$$

$$= \frac{\sqrt{ab}\,(a - 2\sqrt{ab} + b)}{a+b}$$

$$= \frac{\sqrt{ab}\,(\sqrt{a} - \sqrt{b})^2}{a+b} \geq 0$$

$$\Rightarrow \sqrt{ab} - \frac{2}{\frac{1}{a}+\frac{1}{b}} \geq 0 \Rightarrow \sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$$

Q> For $x, y, z \in \mathbb{R}^+$, $(x+y)(y+z)(z+x) \geq 8xyz$

Ans:- HomeWork

Q> For $x, y, z \in \mathbb{R}$, $x^2 + y^2 + z^2 \geq xy + yz + zx$

Ans:- HomeWork